Relay Round
125R00

(1-1) Let $T = 56$. Let K be the number of centimeters in T inches, rounded to the nearest integer. (One inch is exactly 2.54 centimeters.) Find the remainder when $K + 23$ is divided by 100.

(1-2) Let $T = TNYWR$. Let K be the sum of the first T natural numbers. Find the remainder when $K + 41$ is divided by 100.

(1-3) Let $T = TNYWR$. Let $\frac{K\pi}{2}$ be the area of the circle that can be circumscribed about a square of area T^2. Find the remainder when $K + 57$ is divided by 100.

Relay Round
125R00

(2-1) Let $T = 65$. Let K be the number of ways to form a T-member committee from a group of $T+3$ people. Find the remainder when $K+91$ is divided by 100.

(2-2) Let $T = TNYWR$. Let $\frac{K\sqrt{3}}{16}$ be the area of an equilateral triangle with side length $T+1$. Find the remainder when $K+28$ is divided by 100.

(2-3) Let $T = TNYWR$. Let K be the number of positive divisors of $(T+1)^2$. Find the remainder when $K+95$ is divided by 100.

Relay Round
125R00

(3-1) Let $T = 28$. Let K be the area formed by the region enclosed by the lines $y = 10(T+2)x+20(T+2)$, $y = -20(T+2)x + 20(T+2)$, and the x-axis. Find the remainder when $K+12$ is divided by 100.

(3-2) Let $T = TNYWR$. The numbers $\{0, 1, 2, \ldots, 9\}$ are placed in order around a circle. A student puts a counter on the number 0. On the first move, she moves the counter 1^1 steps clockwise to the number 1. On the second move, she moves the counter 2^2 steps clockwise to the number 5. On the third move, she moves the counter 3^3 steps clockwise to the number 2. Let M be where the counter ends up after the $(T+2)$th move. Let $K = (M+4)^3$. Find the remainder when $K+6$ is divided by 100.

(3-3) Let $T = TNYWR$. Let K be the number of rectangles (including squares) of any size that can be formed from the union of squares in a $T \times T$ checkerboard. Find the remainder when $K+51$ is divided by 100.

Relay Round
125R00

(4-1) Let $T = 50$. A $4T \times 8T$ rectangle is inscribed in a semicircle with one side on the diameter of the semicircle. The larger possible area of the semicircle is $K\pi$. Find the remainder when $K + 9$ is divided by 100.

(4-2) Let $T = TNYWR$. Let a, b, and c be complex numbers that satisfy the following system of equations:

$$\begin{aligned} a + b + c &= T \\ ab + ac + bc &= T \\ abc &= T. \end{aligned}$$

Let K be the value of $a^2 + b^2 + c^2 + a + b + c$. Find the remainder when $K + 85$ is divided by 100.

(4-3) Let $T = TNYWR$. Let $\triangle ABC$ have side lengths $AB = T + 2$, $AC = T + 3$, and $BC = T + 4$. Let R be the circumradius and r be the inradius of this triangle. Let $K = 6Rr$. Find the remainder when $K + 26$ is divided by 100.

(5-1) Let $T = 90$. Let K be the greatest number of regions into which T lines can divide the plane. Find the remainder when $K + 8$ is divided by 100.

(5-2) Let $T = TNYWR$. The finite continued fraction expansion of $\frac{7T+5}{3T+2}$ can be written as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

where $a_0, a_1, \ldots, a_n$ are the n integers needed for this expansion for some integer n. Let $K = \sum_{k=0}^{n} |a_k|$. Find the remainder when $K + 74$ is divided by 100.

(5-3) Let $T = TNYWR$. Let a, b, and c be complex numbers that satisfy the following system of equations:

$$\begin{aligned} a + b + c &= T \\ ab + ac + bc &= T \\ abc &= T. \end{aligned}$$

Let K be the value of $a^3 + b^3 + c^3$. Find the remainder when $K + 58$ is divided by 100.

(1-1)	65	(K = 142)	(1-2)	86	(K = 2145)	(1-3)	53	(K = 7396)
(2-1)	7	(K = 50116)	(2-2)	84	(K = 256)	(2-3)	4	(K = 9)
(3-1)	12	(K = 900)	(3-2)	49	(K = 343)	(3-3)	76	(K = 1500625)
(4-1)	9	(K = 85000)	(4-2)	57	(K = 72)	(4-3)	25	(K = 3599)
(5-1)	4	(K = 4096)	(5-2)	83	(K = 9)	(5-3)	27	(K = 551369)

Relay Round
125R00

(1-1) Since 1 in $= 2.54$ cm, K equals $2.54T$ rounded to the nearest integer.

(1-2) The sum of the natural numbers from 1 to n is $\frac{n(n+1)}{2}$, meaning that the sum of the first T natural numbers is $K = \frac{T(T+1)}{2}$.

(1-3) Since the area of the square is given to be T^2, the side length of the square is T and the diagonal length of the square is $T\sqrt{2}$. Since the circle that circumscribes the square has the same center as the square, the radius of the circle will be $\frac{1}{2}$ of the square's diagonal. The area of the circle is $A = \pi\frac{(T\sqrt{2})^2}{2} = \frac{T^2\pi}{2}$. Since $\frac{K\pi}{2}$ is the area of the desired circle, $K = T^2$.

(2-1) The number of T-member committees that can be formed from a group of $T+3$ people is $K = \binom{T+3}{T}$.

(2-2) The area of an equilateral triangle with side length $T+1$ is $\frac{(T+1)^2\sqrt{3}}{4}$. Since this area is equal to $\frac{K\sqrt{3}}{16}$, we set these equations equal and solve for K to get $\frac{K\sqrt{3}}{16} = \frac{(T+1)^2\sqrt{3}}{4}$, so $K = 4(T+1)^2$.

(2-3) Let the prime factorization of $T+1$ be $p^a q^b \cdots$. Then the prime factorization of $(T+1)^2$ is $p^{2a}q^{2b}\cdots$, so the number of prime factors that it has is $(2a+1)(2b+1)\cdots$.

(3-1) We must find the intersection of the three lines that form the points for the enclosed region. Equating the first and second equations and solving, we get that the first intersection point is $(0, 20(T+2))$. Equating the first and third equations and solving, we get that the second intersection point is $(-2, 0)$. Equating the second and third equations and solving, we get that the final intersection point is $(1, 0)$. The enclosed region is a triangle with base length $1-(-2) = 3$ and height $20(T+2)$, so $K = A = \frac{3(20(T+2))}{2} = 30(T+2)$.

(3-2) Realize that the next place where the counter will end up is defined by the function $f(x) = 1^1 + 2^2 + 3^3 + 4^4 + \ldots + x^x \bmod 10$. Taking $x^x \bmod 10$ for $x = 1$ to $x = 20$, we see that the units digits are $1, 4, 7, 6, 5, 6, 3, 6, 9, 0, 1, 6, 3, 6, 5, 6, 7, 4, 9, 0, \ldots$. Realize that for the first 10 remainders and the next 10 remainders that they are the same set of numbers, just in a different order. Their sum is $1+4+7+6+5+6+3+6+9+0 = 47$. We now have to find $M = f(T+2)(\text{mod}10)$. Plugging in the value of T, if $T+2 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \bmod 20$, then $M = 47\left\lfloor(\frac{T+2}{10})\right\rfloor +$(any other residues from the set $\{1, 4, 7, 6, 5, 6, 4, 6, 9, 0\}$) mod10. However, if $T+2 = 11, 12, 13, 14, 15, 16, 17, 18, 19, 0 \bmod 20$, then $M = 47\left\lfloor\frac{T+2}{10})\right\rfloor +$(any other residues from the set $\{1, 6, 3, 6, 5, 6, 7, 4, 9, 0\}$) mod10. Since $K = (M+4)^3$ and M is a digit between 0 and 9, this calculation can be done via brute force after finding M.

(3-3) A rectangle consists of 2 horizontal and 2 vertical lines connected together perpendicularly. This problem reduces to finding the number of ways to choose 2 horizontal and 2 vertical lines such that they form a rectangle. In a $T \times T$ checkerboard, there will be $T+1$ horizontal and $T+1$ vertical grid lines that form the checkerboard, so the number of ways to choose 2 horizontal and 2 vertical lines such that they form a rectangle is $K = \binom{T+1}{2} \cdot \binom{T+1}{2} = \binom{T+1}{2}^2$.

(4-1) There are two cases to consider: either the side length $4T$ or the side length $8T$ lies on the diameter of this semicircle. If the side length $4T$ lies on the diameter of the circle, then the length of the radius would be $r = \sqrt{(2T)^2 + (8T)^2} = \sqrt{4T^2 + 64T^2} = 2T\sqrt{17}$. If the side length $2T$ lies on the diameter of the circle, then the length of the radius would be $r = \sqrt{(4T)^2 + (4T)^2} = 4T\sqrt{2}$. Since $2\sqrt{17} > 4\sqrt{2}$, the rectangle that has side length $4T$ on the semicircle's diameter will produce the largest radius and the largest area. This area is $K = \frac{1}{2}\pi(2T\sqrt{17})^2 = 34T^2\pi$, meaning that $K = 34T^2$.

(4-2) Realize that $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$. Then we have $a^2 + b^2 + c^2 = T^2 - 2T$, so $K = a^2 + b^2 + c^2 + a + b + c = T^2 - T$.

(4-3) Consider two versions of the area of a triangle: $\frac{(AB)(BC)(AC)}{4R}$ and rs, where a, b, and c are the side lengths of $\triangle ABC$, s is the semiperimeter of the triangle, R is the circumradius of the triangle, and r is the inradius of the triangle. Setting these area equations equal to each other, we see that

$$\begin{aligned}
\frac{(AB)(BC)(AC)}{4R} &= r\left(\frac{AB + BC + AC}{2}\right) \\
\frac{(T+2)(T+3)(T+4)}{4R} &= r\left(\frac{T+2+T+3+T+4}{2}\right) \\
\frac{(T+2)(T+3)(T+4)}{4R} &= r\left(\frac{3(T+3)}{2}\right) \\
(T+2)(T+3)(T+4) &= 6Rr(T+3) \\
6Rr &= (T+2)(T+4).
\end{aligned}$$

Therefore, $K = 6Rr = (T+2)(T+4)$.

(5-1) The greatest number of regions into which T lines can divide the plane is $K = \binom{T}{2} + T + 1$.

(5-2) The finite continued fraction expansion of $\frac{7T+5}{3T+2} = 2 + \frac{T+1}{3T+2} = 2 + \frac{1}{\left(\frac{3T+2}{T+1}\right)} = 2 + \frac{1}{2+\frac{T}{T+1}} = 2 + \frac{1}{2+\frac{1}{\frac{T+1}{T}}} =$ $2 + \frac{1}{2+\frac{1}{1+\frac{1}{T}}}$. Therefore, $a_0 = 2$, $a_1 = 2$, and $a_2 = 1$, and $a_3 = T$. Therefore, $K = \sum_{k=0}^{n} |a_k| = 2+2+1+T = T+5$.

(5-3) Realize that $a^3 + b^3 + c^3 = (a+b+c)(a^2+b^2+c^2-(ab+ac+bc)) + 3abc = T(a^2+b^2+c^2-T) + 3T$. From the given equations, we see that $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+ac+bc) = T^2 - 2T$, meaning that $K = a^3 + b^3 + c^3 = T(T^2 - 3T) + 3T = T^3 - 3T^2 + 3T = (T-1)^3 + 1$.

